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# A-4 STEP CHEBYSHEV BASED MULTIDERIVATIVE DIRECT SOLVER FOR THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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## ABSTRACT

This paper develops and examines a uniform order, 4 step block method with Chebyshev series as bases function. Collocation and interpolation technique was used to modeled implicit discrete schemes from continuous scheme to derive our block. The block method obtained was of order 4. It is consistent, zero stable and consequently zero stable. The results obtained from four test problems shown that the method converges to exact solutions and perform better than some existing methods in the literatures.

*Keywords:* Multiderivative, Chebyshev series, Continuous Scheme, Direct solver.

## INTRODUCTION

A problem of the form:

$$y'''=f(x,y,y',y''),\quad y(a)=\alpha,\ y'(a)=\beta,\ y''(a)=\gamma \qquad (1.1)$$

For a continuous differentiable function f is called third order initial value problems of ordinary differential equations. Problem (1.1) is conventionally solved by reducing it to an equivalent system of first order ordinary differential equations in four equations comprising of four dependent variables $y_1, y_2$ and $y_3$ in the form:

$$y_1'=f_1(x,y_1,y_2,y_3),\quad y_1(a)=\eta_1$$
$$y_2'=f_2(x,y_1,y_2,y_3),\quad y_2(a)=\eta_2$$
$$y_3'=f_3(x,y_1,y_2,y_3),\quad y_3(a)=\eta_3 \qquad (1.2)$$

Method of reducing problem of kind (1.1) to the form (1.2) is extensively discussed by Awoyemi (1992), Jennings (1987), and Jator (2001, 2007) among other many authors. There are major

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drawbacks associated with method that solve equivalent system of first order ordinary differential equations (1.2) such as the problem of writing computer subroutine - sub program within the main program to get starting values, extra computational effort, more computer time and storage wastage. To circumvent the aforementioned problems, several researchers have developed several methods for direct solution of higher order ordinary differential equations. Some of these authors have adopted different polynomials including Legendre Polynomial (Ademola ,2017) Power series (Familua et-al, 2017), Lucas Polynomia( Adeniran and Longe, 2019), Taylor series (Adoghe et-al, 2016), and Chebyshev (Kayode et-al , 2018) into developing various block methods for solving problem (1.1) directly. They developed high order method to handle high order ordinary differential equations.

An Implicit Hybrid Block Methods for solving fourth order Ordinary differentials Equations was developed by Kayode, Duromola and Bolarinwa (2014). Adeniran and Longe (2019), Guler et al (2019), Ramos et al (2020) and Singla et al (2021) have developed direct methods for solving second order initial value problems. Adeniran and Zurni (2019) developed a three- step implicit hybrid solver for third order initial value problems. This paper is developed and examines a block method for the solution of fourth order initial value problems using the Chebyshev polynomial as basis function.

**FORMULATION OF THE METHOD**

In this paper, we developed a 4- step block method through interpolation and collocation approach. The Chebyshev polynomial is used as basis function. In order to solve equation (1.1), we employ the approximate solution.

$$y(x) = \sum_{r=0}^{t+s-1} a_r T_r(x) \qquad (2.1)$$

where t and s are number of interpolation and collocation respectively on the partition a=x0x1≤⋯xk=b of the integration interval a,b .

TrX is the nth degree Chebyshev polynomial of the first kind defined as

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$$T_r(x) = \cos\left[r\cos^{-1}\left(\frac{2x-(b+a)}{b-a}\right)\right] \tag{2.2}$$

The recurrence relation for the Chebyshev series is given as follows:

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x), \quad r \geq 1 \tag{2.3}$$

where T0x=1 and T1x=x.

To interpolate at x=xn+i, i=0,1,3 and collocate at x=xn+i, i=014, equation (2.1) becomes:

$$y(x) = a_0 + a_1\left[\frac{x-kh-2h}{2h}\right] + a_2\left[2\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right] + a_3\left[4\left(\frac{x-kh-2h}{2h}\right)^3 - 3\left(\frac{x-kh-2h}{2h}\right)\right] + a_4\left[8\left(\frac{x-kh-2h}{2h}\right)^4 - 8\left(\frac{x-kh-2h}{2h}\right)^2 + 1\right] + a_5\left[16\left(\frac{x-kh-2h}{2h}\right)^5 - 20\left(\frac{x-kh-2h}{2h}\right)^3 + 5\left(\frac{x-kh-2h}{2h}\right)\right] + a_6\left[32\left(\frac{x-kh-2h}{2h}\right)^6 - 48\left(\frac{x-kh-2h}{2h}\right)^4 + 18\left(\frac{x-kh-2h}{2h}\right)^2 - 1\right] + a_7\left[64\left(\frac{x-kh-2h}{2h}\right)^7 - 112\left(\frac{x-kh-2h}{2h}\right)^5 + 56\left(\frac{x-kh-2h}{2h}\right)^3 - 7\left(\frac{x-kh-2h}{2h}\right)\right] \tag{2.4}$$

First, second and third derivatives of (2.4) are obtained as follows:

$$4h^2y''(x) = a_2[4] + a_3\left[24\left(\frac{x-kh-2h}{2h}\right)\right] + a_4\left[96\left(\frac{x-kh-2h}{2h}\right)^2 - 16\right] + a_5\left[320\left(\frac{x-kh-2h}{2h}\right)^3 - 120\left(\frac{x-kh-2h}{2h}\right)\right] + a_6\left[960\left(\frac{x-kh-2h}{2h}\right)^4 - 576\left(\frac{x-kh-2h}{2h}\right)^2 + 36\right] + a_7\left[2688\left(\frac{x-kh-2h}{2h}\right)^5 - 2240\left(\frac{x-kh-2h}{2h}\right)^3 + 336\left(\frac{x-kh-2h}{2h}\right)\right] \tag{2.6}$$

$$2hy'(x) = a_1 + a_2\left[4\left(\frac{x-kh-2h}{2h}\right)\right] + a_3\left[12\left(\frac{x-kh-2h}{2h}\right)^2 - 3\right] + a_4\left[32\left(\frac{x-kh-2h}{2h}\right)^3 - 16\left(\frac{x-kh-2h}{2h}\right)\right] + a_5\left[80\left(\frac{x-kh-2h}{2h}\right)^4 - 60\left(\frac{x-kh-2h}{2h}\right)^2 + 5\right] + a_6\left[192\left(\frac{x-kh-2h}{2h}\right)^5 - 192\left(\frac{x-kh-2h}{2h}\right)^3 + 36\left(\frac{x-kh-2h}{2h}\right)\right] + a_7\left[448\left(\frac{x-kh-2h}{2h}\right)^6 - 560\left(\frac{x-kh-2h}{2h}\right)^4 + 168\left(\frac{x-kh-2h}{2h}\right)^2 - 7\right] \tag{2.5}$$

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and

$$8h^3y'''(x) = a_3[24] + a_4\left[192\left(\frac{x-kh-2h}{2h}\right)\right] + a_5\left[960\left(\frac{x-kh-2h}{2h}\right)^2 - 120\right] + a_6\left[3840\left(\frac{x-kh-2h}{2h}\right)^3 - 1152\left(\frac{x-kh-2h}{2h}\right)\right] + a_7\left[13440\left(\frac{x-kh-2h}{2h}\right)^4 - 6720\left(\frac{x-kh-2h}{2h}\right)^2 + 336\right] \quad (2.7)$$

Interpolating (2.4) at x=xn,xn+1 and xn+3 and collocating (2.7) at x=xn,xn+1,xn+2,xn+3 and xn+4 lead to the matrix equation:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} & -1 & -\frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 3 & -24 & 105 & -336 & 882 \\ 0 & 0 & 0 & 3 & -12 & 15 & 12 & -63 \\ 0 & 0 & 0 & 3 & 0 & -15 & 0 & 42 \\ 0 & 0 & 0 & 3 & 12 & 15 & -12 & -63 \\ 0 & 0 & 0 & 3 & 24 & 105 & 336 & 882 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+3} \\ h^3 f_n \\ h^3 f_{n+1} \\ h^3 f_{n+2} \\ h^3 f_{n+3} \\ h^3 f_{n+4} \end{pmatrix} \quad (2.8)$$

Solving (2.8) in the unknown parameters ai, i=017 leads to:

$$a_0 = \frac{23}{7560}h^3 f_n + \frac{559}{7560}h^3 f_{n+1} + \frac{89}{504}h^3 f_{n+2} + \frac{601}{7560}h^3 f_{n+3} + \frac{1}{3780}h^3 f_{n+4} + \frac{1}{3}y_n + \frac{2}{3}y_{n+3},$$

$$a_1 = \frac{1}{720}h^3 f_n + \frac{13}{90}h^3 f_{n+1} + \frac{3}{8}h^3 f_{n+2} + \frac{13}{90}h^3 f_{n+3} + \frac{1}{720}h^3 f_{n+4} + y_{n+3} - y_{n+1},$$

$$a_2 = \frac{13}{3024}h^3 f_n + \frac{733}{3780}h^3 f_{n+1} + \frac{221}{630}h^3 f_{n+2} + \frac{439}{3780}h^3 f_{n+3} + \frac{23}{15120}h^3 f_{n+4} + \frac{2}{3}y_n - y_{n+1} + \frac{1}{3}y_{n+3},$$

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$$a_3 = \frac{7}{40}h^3 f_{n+2} + \frac{7}{90}h^3 f_{n+1} + \frac{7}{90}h^3 f_{n+3} + \frac{1}{720}h^3 f_n + \frac{1}{720}h^3 f_{n+4},$$

$$a_4 = -\frac{7}{180}h^3 f_{n+1} + \frac{7}{180}h^3 f_{n+3} - \frac{1}{720}h^3 f_n + \frac{1}{720}h^3 f_{n+4},$$

$$a_5 = \frac{1}{90}h^3 f_{n+1} - \frac{1}{40}h^3 f_{n+2} + \frac{1}{90}h^3 f_{n+3} + \frac{1}{720}h^3 f_n + \frac{1}{720}h^3 f_{n+4},$$

$$a_6 = -\frac{1}{720}h^3 f_n + \frac{1}{360}h^3 f_{n+1} - \frac{1}{360}h^3 f_{n+3} + \frac{1}{720}h^3 f_{n+4},$$

and

$$a_7 = \frac{1}{2520}h^3 f_n - \frac{1}{630}h^3 f_{n+1} + \frac{1}{420}h^3 f_{n+2} - \frac{1}{630}h^3 f_{n+3} + \frac{1}{2520}h^3 f_{n+4} \qquad (2.9)$$

Substituting (2.9) into equations (2.4) to (2.6) and evaluating the resulting continuous scheme for (2.4) at x=xn+2 and x=xn+4 and other continuous schemes at x=xn,xn+1,xn+2,xn+3,xn+4 respectively lead to:

$$y_{n+3} + 3y_{n+2} - 3y_{n+1} + y_n = \frac{h^3}{5040}(-19f_n - 2444f_{n+1} - 2634f_{n+2} + 76f_{n+3} - 19f_{n+4}) \qquad (2.10)$$

$$y_{n+4} - 2y_{n+3} + 2y_{n+1} - y_n = \frac{h^3}{2520}(23f_n + 1168f_{n+1} + 2658f_{n+2} + 1168f_{n+3} + 23f_{n+4}) \qquad (2.11)$$

$$y_{n+3} - 9y_{n+1} + 8y_n + 6hy_n' = \frac{h^3}{5040}(1861f_n + 11960f_{n+1} + 582f_{n+2} + 872f_{n+3} - 155f_{n+4}) \qquad (2.12)$$

$$-y_{n+3} - 3y_{n+1} + 4y_n + 6hy'_{n+1} = \frac{h^3}{1260}(-64f_n - 1760f_{n+1} - 699f_{n+2} + 4f_{n+3} - f_{n+4}) \qquad (2.13)$$

$$-y_{n+3} + y_{n+1} + 2hy'_{n+2} = \frac{h^3}{720}(f_n - 16f_{n+1} - 210f_{n+2} - 16f_{n+3} + f_{n+4}) \qquad (2.14)$$

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$$-5y_{n+3}+9y_{n+1}-4y_n+6hy'_{n+3}=\frac{h^3}{1260}(43f_n+2348f_{n+1}+4605f_{n+2}+584f_{n+3}-20f_{n+4}) \quad (2.15)$$

$$-7y_{n+3}+15y_{n+1}-8y_n+6hy'_{n+4}=\frac{h^3}{5040}(197f_n+1962f_{n+1}+43014f_{n+2}+30712f_{n+3}+2213f_{n+4}) \quad (2.16)$$

$$-4y_{n+3}+12y_{n+1}-8y_n+12h^2y''_{n}=\frac{h^3}{630}(-2456f_n-7585f_{n+1}+636f_{n+2}-323f_{n+3}+148f_{n+4}) \quad (2.17)$$

$$-4y_{n+3}+12y_{n+1}-8y_n+12h^2y''_{n+1}=\frac{h^3}{1260}(359f_n-1604f_{n+1}-4272f_{n+2}+580f_{n+3}-103f_{n+4}) \quad (2.18)$$

$$-4y_{n+3}+12y_{n+1}-8y_n+12h^2y''_{n+2}=\frac{h^3}{630}(-20f_n+283f_{n+1}+2652f_{n+2}-487f_{n+3}+64f_{n+4}) \quad (2.19)$$

$$-4y_{n+3}+12y_{n+1}-8y_n+12h^2y''_{n+3}=\frac{h^3}{1260}(191f_n+4108f_{n+1}+14880f_{n+2}+6292f_{n+3}-271f_{n+4}) \quad (2.20)$$

$$-4y_{n+3}+12y_{n+1}-8y_n+12h^2y''_{n+4}=\frac{h^3}{630}(-104f_n+3167f_{n+1}+4668f_{n+2}+9929f_{n+3}+2500f_{n+4}) \quad (2.21)$$

Equations (2.10) – (2.21) combined constitute our block method.

## ANALYSIS OF THE METHOD

Basic properties of the block method are considered, defined and analyzed to establish the efficiency and reliability of the method. The following properties are analyzed: Order, error constant, consistence and Zero stability. Establishing the last two properties guarantee the convergence of the method.

### 3.1 Local Truncation Error

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According to Fatunla (1988) and Lambert (1973), the local truncation error associated with the third order linear multistep method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h^3 \sum_{i=0}^{k} \beta_i f_{n+i} \qquad (3.1)$$

is defined by the difference operator

$$L[y(x):h] = \sum_{i=0}^{k} [\alpha_i y(x_n + ih) - h^3 \beta_i f(x_n + ih)], \qquad (3.2)$$

where y(x) is an arbitrary function, continuously differentiable on [a, b].

Expanding (3.2) in Taylor series about point x leads to the expression:

$$L[y(x):h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \cdots + c_p h^p y^{(p)}(x) + \cdots c_{p+3} h^{p+3} y^{(p+3)}(x) \; (3.3)$$

where c0,c1,c2,, cp,,cp+2 are obtained as follow:

$$c_0 = \sum_{i=0}^{k} \alpha_i$$

$$c_1 = \sum_{i=1}^{k} i\alpha_i$$

$$c_2 = \frac{1}{2!} \sum_{i=1}^{k} i^2 \alpha_i$$

$$\vdots$$

$$c_q = \frac{1}{q!} \left[ \sum_{i=1}^{k} i^q \alpha_i - q(q-1)(q-2) \sum_{i=1}^{k} \beta_i i^{q-3} \right]$$

Therefore, method (3.1) is of order P if

c0=c1=c2==cp=cp+1=cp+2=0 and cp+30.

The constant cp+30 is called the error constant and cp+3h$^{p+3}$y$^{p+3}$(x) is the principal local truncation error at xn.

Using the above definition, the block method (2.10 - 2.21) is of order P=4 and error

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$$\text{constant} = \left[\frac{1}{2520}, \frac{1}{1260}, \frac{1}{2520}, \frac{1}{2520}, \frac{1}{2520}, \frac{1}{504}, \frac{1}{360}, \frac{1}{630}, \frac{1}{630}, \frac{1}{630}, \frac{1}{630}, \frac{1}{630}\right]^T.$$

**3.2 Consistency**

The linear multistep method (3.1) is said to be consistent if it satisfies the following conditions:

(i) The order $P \geq 1$

(ii) $\sum_{i=0}^{k} \alpha_i = 0$

(iii) $\rho(1) = \rho'(1) = 0$ and

(iv) $\rho'''(1) = 3!\,\sigma(1)$

Where r and r are the first and second characteristic polynomials. Condition (i) is sufficient for the associated block method to be consistent. (Jator, 2007).

The main method (2.10) satisfies condition (i)- (iv) above. Therefore, the method is consistent.

**3.3 Zero Stability**

To analyze the method for zero stability, we write equations (2.10) - (2.21) in the matrix equation:

$\alpha = 3$,

$Y_m = \left(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y'_{n+1}, y'_{n+2}, y'_{n+3}, y'_{n+4}, y''_{n+1}, y''_{n+2}, y''_{n+3}, y''_{n+4}\right)^T$,

$Y_{m-1} = \left(y_{n-3}, y_{n-2}, y_{n-1}, y_n, y'_{n-3}, y'_{n-2}, y'_{n-1}, y'_n, y''_{n-3}, y''_{n-2}, y''_{n-1}, y''_n\right)^T$

$F_m = \left(f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}\right)^T$ .

$$AY_m = BY_{m-1} + h^{\alpha}[Cf_n + DF_m], \quad (3.4)$$

Where

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$$\begin{pmatrix} -3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -1 & 0 & 6h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 2h & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 0 & -5 & 0 & 0 & 0 & 6h & 0 & 0 & 0 & 0 & 0 \\ 15 & 0 & -7 & 0 & 0 & 0 & 0 & 6h & 0 & 0 & 0 & 0 \\ 12 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 12h^2 & 0 & 0 & 0 \\ 12 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 12h^2 & 0 & 0 \\ 12 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12h^2 & 0 \\ 12 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12h^2 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \\ y''_{n+1} \\ y''_{n+2} \\ y''_{n+3} \\ y''_{n+4} \end{pmatrix}$$

Consequently;

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12h^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \\ y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \\ y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y'' \end{pmatrix} +$$

$$h^3 \begin{pmatrix} \frac{-19}{5040} \\ \frac{23}{2520} \\ \frac{1861}{5040} \\ \frac{-64}{1260} \\ \frac{1}{720} \\ \frac{43}{1260} \\ \frac{197}{5040} \\ \frac{-2456}{630} \\ \frac{359}{1260} \\ \frac{-20}{630} \\ \frac{191}{1260} \\ \frac{-104}{630} \end{pmatrix} (f_n) + \begin{pmatrix} \frac{-2444}{5040} & \frac{-2634}{5040} & \frac{76}{5040} & \frac{-19}{5040} \\ \frac{1168}{2520} & \frac{2658}{2520} & \frac{1168}{2520} & \frac{23}{2520} \\ \frac{11960}{5040} & \frac{582}{5040} & \frac{872}{5040} & \frac{-155}{5040} \\ \frac{-1760}{1260} & \frac{-699}{1260} & \frac{4}{1260} & \frac{-1}{1260} \\ \frac{-16}{720} & \frac{-210}{720} & \frac{-16}{720} & \frac{1}{720} \\ \frac{2348}{1260} & \frac{4605}{1260} & \frac{584}{1260} & \frac{-20}{1260} \\ \frac{19624}{5040} & \frac{43014}{5040} & \frac{30712}{5040} & \frac{2213}{5040} \\ \frac{-7585}{630} & \frac{636}{630} & \frac{-823}{630} & \frac{148}{630} \\ \frac{-1604}{1260} & \frac{-4272}{1260} & \frac{580}{1260} & \frac{-103}{1260} \\ \frac{283}{630} & \frac{2652}{630} & \frac{-487}{630} & \frac{64}{630} \\ \frac{4108}{1260} & \frac{14880}{1260} & \frac{6292}{1260} & \frac{-271}{1260} \\ \frac{3167}{630} & \frac{4668}{630} & \frac{9929}{630} & \frac{2500}{630} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \qquad (3.5)$$

The normalized version of (3.4) is given by

$$A^* Y_m = B^* Y_{n-1} + h^3 [C^* f_n + D^* F_m] , \qquad (3.6)$$

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Hence, the zero stability of the method is determined by the expression:

$$\rho(r) = det(rA^* - B^*) = 0 \text{ as } h \rightarrow 0 \qquad (3.7)$$

Such that :

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$$A^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

:Solving for r in equation (3.7), $r=r^{11}r-1=0 \Rightarrow r=0,1$. therefore, the method is zero stable.

### 3.4 Convergence

According to Dahlquist (1994), a method converges if it is consistent and zero stable. Thus, the block method is convergent since it is consistent and zero stable.

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**NUMERICAL EXAMPLES**

To investigate the efficiency of the method, we apply the method to solve two test problems.

**Example 4.1**

Solve the initial value problem:

$$y''' + y' = 0 \; ; \; y(0) = 0, \, y'(0) = 1, \, y''(0) = 2. \quad h = 0.1$$

Analytical Solution is yx=2(1-cos x)+sin x.

Source: Anake et-al (2013).

**Example 4.2**

Solve the initial value problem:

$$y''' - y'' + y' - y = 0 \; ; \; y(0) = 1, \, y'(0) = 0, \, y''(0) =- 1. \quad h = 0.01$$

Analytical Solution is x=cos x .

Source: Kuboye et-al (2020).

**Example 4.3**

Consider the initial value problem:

$$y''' - \exp(x) = 0 \; ; \; y(0) = 3, \, y'(0) = 1, \, y''(0) = 5. \quad h = 0.1$$

Analytical Solution: $y(x)=2+2x^2+\exp(x)$.

Source: Olabode (2013)., Obarhua and Kayode (2016).

**Example 4.4**

Consider the initial value problem:

$$y''' - 3\cos x = 0 \; ; \; y(0) = 1, \, y'(0) = 0, \, y''(0) = 2. \quad h = 0.05$$

Analytical Solution: $y(x)=x^2+3x-3\sin x+1$

Source: Obarhua and Kayode (2016).

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Absolute errors in the solution of Example (4.1) - (4.4) base on our method are respectively presented in Table (4.1) - (4.4) in comparison with existing methods in literature.

**Table 4.1. Absolute errors for Example 4.1 with h= 0.1**

| x | Exact | New methods | Error in new method | Error in Awoyemi Order 6 | Error in Olabode Order 8 | Error in Anake et-al. Order 4 |
|---|---|---|---|---|---|---|
| 0.1 | 0.109825086 | 0.109825086 | 6.19E-11 | - | 1.67E-08 | 1.61E-09 |
| 0.2 | 0.238536175 | 0.238536176 | 3.98E-10 | 8.85E-07 | 3.81E-07 | 1.04E-08 |
| 0.3 | 0.384847228 | 0.384847229 | 1.02E-09 | - | 1.57E-07 | 2.96E-08 |
| 0.4 | 0.547296354 | 0.547296356 | 1.80E-09 | 6.69E-06 | 3.99E-06 | 2.31E-07 |
| 0.5 | 0.724260414 | 0.724260418 | 2.73E-09 | - | 7.96E-06 | 4.54E-07 |
| 0.6 | 0.913971243 | 0.913397125 | 3.82E-09 | 2.37E-05 | 1.37E-05 | 1.47E-06 |
| 0.7 | 1.114533313 | 1.114533178 | 5.11E-09 | - | 2.12E-05 | 2.87E-06 |
| 0.8 | 1.323942672 | 1.323942678 | 6.38E-09 | 5.52E-05 | 3.04E-05 | 4.68E-06 |
| 0.9 | 1.540106973 | 1.540106981 | 7.73E-09 | - | 4,10E-05 | 6.92E-06 |
| 1.0 | 1.760866373 | 1.760866382 | 9.09E-09 | - | - | - |

**Table 4.2. Absolute errors for Example 4.2 with h= 0.01**

| x | Exact | New method | Error in new method | Error in Badmus & Yaya (2009) Order 5 | Adeniyi & Mohammed (2014) Order 5 | Error in Ademola (2017) Order 6 | Error in Kuboye et-al Order 6 |
|---|---|---|---|---|---|---|---|
| 0.01 | 0.9999500004 | 0.9999500004 | 3.5E-19 | 9.80E-08 | 6.72E-07 | 1.25E-09 | 1.11E-16 |
| 0.02 | 0.9998000067 | 0.9998000067 | 2.24E-18 | 3.95E-07 | 1.34E-06 | 6.68E-09 | 5.55E-16 |
| 0.03 | 0.9995500337 | 0.9995500337 | 5.57E-18 | 8.87E-07 | 2.02E-06 | 1.14E-08 | 8.66E-15 |

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| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.04 | 0.9992001067 | 0.9992001067 | 1.03E-17 | 1.57E-06 | 2.69E-06 | 2.04E-08 | 6.48E14 |
| 0.05 | 0.9987502604 | 0.9987502604 | 1.59E-17 | 2.43E-06 | 3.36E-06 | 9.39E-08 | 2.63E-14 |
| 0.06 | 0.9982005399 | 0.9982005399 | 2.31E-17 | 3.45E-06 | - | 2.67E-07 | - |
| 0.07 | 0.997551003 | 0.9975510003 | 3.19E-17 | 4.60E-06 | - | 5.44E-07 | - |
| 0.08 | 0.9968017063 | 0.9968017063 | 4.19E-17 | 5.84E-06 | - | 1.01E-07 | - |
| 0.09 | 0.995952733 | 0.995952733 | 5.29E-17 | 7.16E-06 | - | 1.66E-06 | - |
| 0.10 | 0.995004165 | 0.995004165 | 6.56E-17 | - | - | - | - |

**Table 4.3. Absolute errors for Example 4.3 with h= 0.1**

| x | Exact | New methods | Error in new method | Error in Olabode | Error in Obarhua & Kayode |
|---|---|---|---|---|---|
| 0.1 | 3.12517092 | 3.12517092 | 0.0 | 9.24E-10 | 4.66E-11 |
| 0.2 | 3.30140276 | 3.30140276 | 0.0 | 18.40E-10 | 4.23E-10 |
| 0.3 | 3.52985881 | 3.52985881 | 0.0 | 24.24E-10 | 1.512E-09 |
| 0.4 | 3.81182470 | 3.81182470 | 0.0 | 53.59E-10 | 3.74E-09 |
| 0.5 | 4.14872127 | 4.14872127 | 0.0 | 7.00E-10 | 1.352E-08 |
| 0.6 | 4.54211880 | 4.54211880 | 0.0 | 3.91E-10 | 1.352E-08 |
| 0.7 | 4.99375271 | 4.99375271 | 0.0 | 6.53E-09 | 2.22E-08 |
| 0.8 | 5.50554093 | 5.50554094 | 1.00E-08 | 2.15E-08 | 3.41E-08 |
| 0.9 | 6.07960311 | 6.07960313 | 2.00E-08 | 3.88E-08 | 5.01E-08 |
| 1.0 | 6.71828183 | 6.71828185 | 2.00E-08 | 6.15E-08 | 7.09E-08 |

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**Table 4.4. Absolute errors for Example 4.4 with h= 0.05**

| x | Exact | New methods | Error in new method | Error in Yahayah and Badmus | Error in Agam and Irhebbhude | Error in Error in Obarhua & Kayode |
|---|---|---|---|---|---|---|
| 0.05 | 1.002562492 | 1.002562492 | 3.46E-14 | - | - | - |
| 0.10 | 1.010499750 | 1.010499750 | 2.24E-13 | 1.04E-06 | 6.40E-08 | 1.38E-11 |
| 0.15 | 1.024185603 | 1.024185603 | 3.82E-13 | - | - | - |
| 0.20 | 1.043992008 | 1.043992008 | 1.01E-12 | 5.06E-06 | 1.26E-07 | 1.89E-09 |
| 0.25 | 1.070288122 | 1.070288122 | 1.60E-12 | - | - | - |
| 0.30 | 1.103439380 | 1.103439380 | 2.64E-12 | 1.21E-005 | 1.52E-07 | 1.71E-08 |
| 0.35 | 1.143806578 | 1.143806578 | 3.18E-12 | - | - | - |
| 0.40 | 1.191744973 | 1.191744973 | 5.96E-12 | 2.22E-05 | 2.13E-07 | 6.91E-08 |
| 0.45 | 1.247603398 | 1.247603398 | 8.06E-12 | - | - | - |
| 0.50 | 1.311723384 | 1.311723384 | 1.09E-11 | 3.53E-05 | 2.73E-07 | 1.94E-07 |

## DISCUSSION

The method developed performed well as the absolute errors obtained are small relatively to error obtainable in methods from the literatures. The method developed which is of order 4 performed better than that of Anake (order 4), Awoyemi (order 6) and Olabode (order 8) in the first test problem. While in the test problem 2, the block method performed better that those of Adeniyi and Mohammed (order 5), Badmus & Yahyah (order 6) and Ademola (order 6). In the exponential problem 4.3, the block method also competes favourably with methods in the literatures. The superiority of the new method is clearly seen in computed absolute errors in comparison with the other three methods in the linear test problem 4.2 shown in Table 4.2 and other three methods in trigonometric test problem 4.4 shown in Table 4.4.

## CONCLUSION

In this paper, we have constructed a direct 4 –step multiderivative integrator which is efficient and suitable for solving third order ordinary differential equations. The method has shown acceptable

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solution and the method performed better than some existing methods. The method behaves well in the four test problems experimented as clearly shown in Table 4.1 - 4.4.

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